Let's Recall what we are doing:

- Want to compute $\iint_D f(x,y,z)\,dx\,dy$

  Example: want $A(D) = \iint_D dx\,dy$

- Too hard in some cases

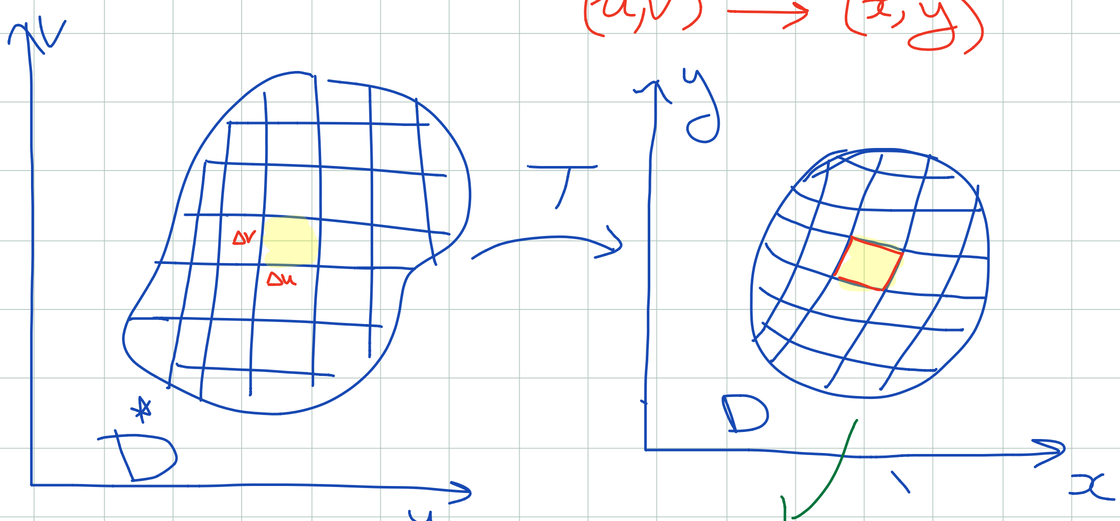
- So we do a change of variable

Example: $x = r\cos\theta$, $y = r\sin\theta$

So $(r,\theta) \longrightarrow (x,y) = (r\cos\theta, r\sin\theta)$

More generally use $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$(u,v) \longrightarrow (x,y)$



We want to compute, say, the area $A(D)$

The Idea

A(D) = sum of areas of the little "almost parallelograms"

$\approx$ sum of areas of parallelograms obtained by linear approx. of T

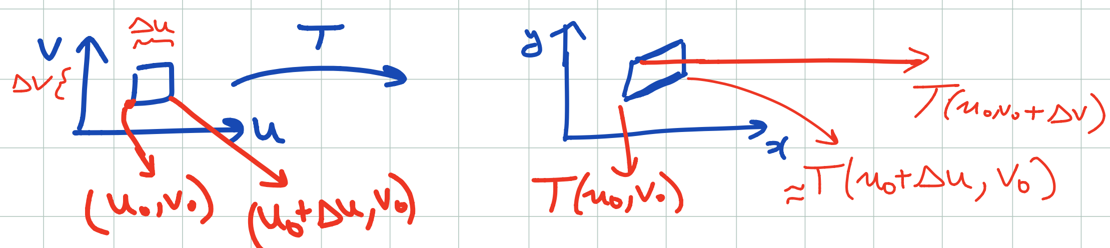
approximately

$\xrightarrow{\text{limit as parallelograms become really small}}$ $A(D) = \iint_D dx\,dy$

So we need the areas of the parallelograms

Linear approximation:

$$T(u,v) \approx T(u_0,v_0) + T'\Big|_{(u_0,v_0)} \begin{pmatrix} u-u_0 \\ v-v_0 \end{pmatrix}$$



$\bullet\, T(u_0+\Delta u, v_0) \approx T(u_0,v_0) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}_{(u_0,v_0)} \begin{bmatrix} \Delta u \\ 0 \end{bmatrix} = T(u_0,v_0) + \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix}_{(u_0,v_0)} \Delta u$

Similarly, $\bullet\, T(u_0, v_0+\Delta v) \approx T(u_0+v_0) + \begin{pmatrix} \partial x/\partial v \\ \partial y/\partial v \end{pmatrix}_{(u_0,v_0)} \Delta v$

& $\bullet\, T(u_0+\Delta v, v_0+\Delta v) = T(u_0,v_0) + \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \end{pmatrix} \Delta u + \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \end{pmatrix} \Delta v$

So the sides of the parallelogram are the vectors

$\frac{\partial x}{\partial u} \Delta u \vec{i} + \frac{\partial y}{\partial u} \Delta u \vec{j}$

and

$\frac{\partial x}{\partial v} \Delta v \vec{i} + \frac{\partial y}{\partial v} \Delta v \vec{j}$

$\to$ sides of parallelogram

Area of the parm $= \begin{vmatrix} \frac{\partial x}{\partial u}\Delta u & \frac{\partial x}{\partial v}\Delta v \\ \frac{\partial y}{\partial u}\Delta u & \frac{\partial y}{\partial v}\Delta v \end{vmatrix} \leftarrow$ determinant

$$= \left[\frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right] \Delta u \Delta v$$

$$= \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v$$

Now, we're ready to find the formula for $A(D)$

$$A(D) \approx \sum_{\text{all }(u_0,v_0)}\sum \text{ areas of little parallelograms}$$

$$= \sum_{\text{all }(u_0,v_0)}\sum \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right|_{(u_0,v_0)} \Delta u \Delta v$$

$$\xrightarrow[\text{shrink}]{\text{as rect.}} \iint_{D^*} \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| du\, dv$$

$$\Rightarrow \quad A(D) = \iint_{D^*} \left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right| du\, dv$$

In General the change of variable formula is

$$\boxed{\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(x(u,v),y(u,v)) \underbrace{\left| \begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix} \right|}_{\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \text{Jacobian Determinant}} du\, dv}$$

$D = T(D^*)$

<u>Example</u>: Use Polar Coordinates to find

$$\iint_{\underbrace{x^2+y^2\leq 1}_{D}} e^{x^2+y^2}\,dx\,dy$$

Sol'n: ① Find the Jacobian det.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\Rightarrow \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| := \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta = r.$$

② Find D^*

$$x^2+y^2 \leq 1 \longleftrightarrow \underbrace{0 \leq r \leq 1 \;\&\; 0 < \theta \leq 2\pi}_{D^*}$$

③ substitute & evaluate

$$\iint_{x^2+y^2\leq 1} e^{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_0^1 e^{r^2}\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 e^{r^2} r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2} e^{r^2}\Big|_{r=0}^{1} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} e - 1\,d\theta = \pi(e-1)$$

In general, polar coordinates change of variable formula:

$$\iint_D f(x,y)\,dx\,dy = \iint_{D^*} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

Change of Variables Formula for triple integrals

Let T be a function $\mathbb{R}^3 \longrightarrow \mathbb{R}^3$ with $x = x(u,v,w)$, $y = y(u,v,w)$ & $z = z(u,v,w)$.

Then the Jacobian determinant of T,

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}}_{\text{Volume of parallelpiped}}$$

$$\iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f\big(x(u,v,w), y(u,v,w), z(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Change of variable formula (3 variable case)

Two Important cases: cylindrical & Spherical coordinates

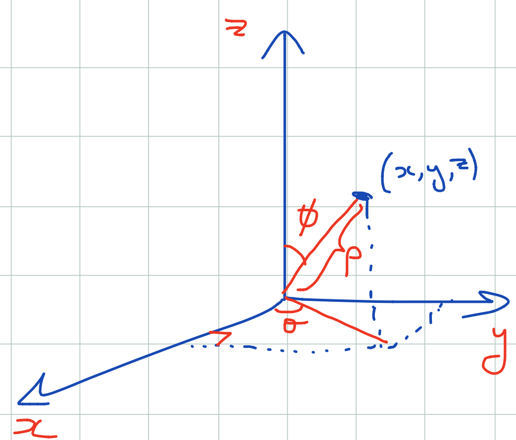
(I) Cylindrical coordinates : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\left|\frac{\partial(x,y,z)}{\partial(r,\theta,z)}\right| = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

So $\displaystyle \iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz$

(II) Spherical coordinates

$$x = \rho\cos\theta\sin\phi$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$



$$\left|\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix}$$

This is the det. of a 3×3 matrix

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei-fh)-b(di-gf)+c(dh-ge)$$

In our case $\left|\dfrac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}\right| = \cdots = \rho^2\sin\phi$

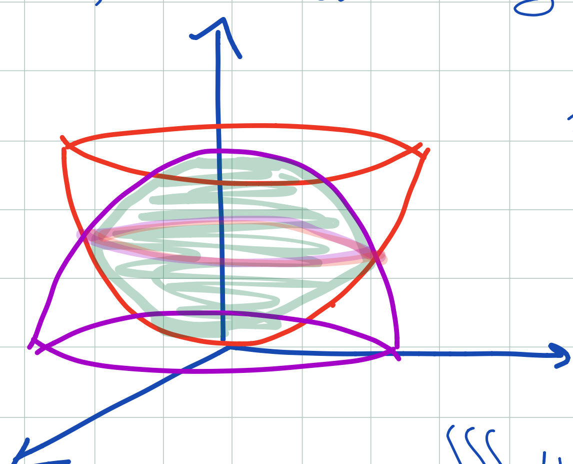
So: $\displaystyle \iiint_W f(x,y,z)\,dx\,dy\,dz = \iiint_{W^*} f(\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)\,\rho^2\sin\phi\; d\rho\,d\theta\,d\phi$

Example Find the volume of the ball of radius R and center $(0,0,0)$ in $\mathbb{R}^3$.

$$V(\text{Ball}) = \iiint_{\text{Ball}} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi = \int_0^{\pi}\int_0^{2\pi}\int_0^{R} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

$$= \int_0^{\pi}\int_0^{2\pi} \frac{R^3}{3} \sin\phi \, d\theta \, d\phi = \int_0^{\pi} \frac{2\pi R^3}{3} \sin\phi \, d\phi = -\frac{2\pi}{3} R^3 (\cos\pi - \cos 0) = \frac{4}{3}\pi R^3$$

Example: Compute the volume of the solid W between the paraboloids $z = x^2+y^2$ & $z = 1-x^2-y^2$



What to use, spherical or cylindrical?

$$\iiint_W dV = \iiint_W r \, dr \, d\theta \, dz$$

$$= \int_0^{2\pi}\int_0^{1/\sqrt{2}}\int_{r^2}^{1-r^2} r \, dz \, dr \, d\theta$$

$r^2 = z$ & $1-r^2 = z$

intersect at $r^2 = 1-r^2$

i.e. $r = 1/\sqrt{2}$

$$= \iint_0^{1/\sqrt{2}} \underbrace{(1-r^2-r^2) r}_{r-2r^3} \, dr \, d\theta$$

$$= \int_0^{2\pi} \frac{r^2}{2} - \frac{r^4}{2} \Big|_0^{1/\sqrt{2}} d\theta = \int_0^{2\pi} \frac{1}{4} - \frac{1}{8} \, d\theta = \frac{\pi}{4}$$

4.3 Vector Fields

Vector field: a vector field $\vec{F}$ is a map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$ that assigns to each point $P=(x_1,\dots,x_n)$ a vector $\vec{F}(P)$ $= \vec{F}(x_1, x_2, \dots, x_n)$.

e.g. velocity fields (wind, fluids)

Force fields (magnetic, gravitational)

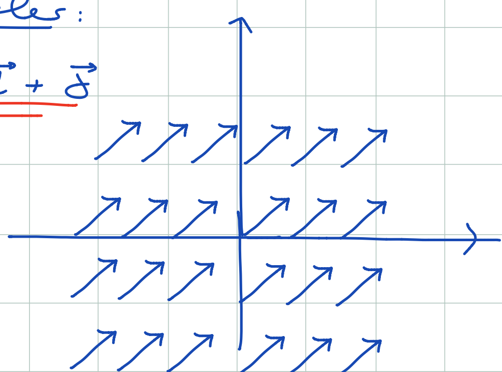
in 2D: $\vec{F} = M\vec{i} + N\vec{j}$

$M(x,y,z)$ $\quad N(x,y,z)$

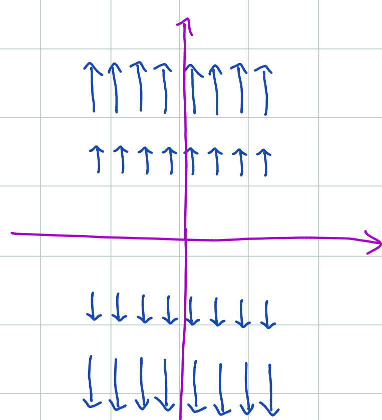
3D: $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$

examples:

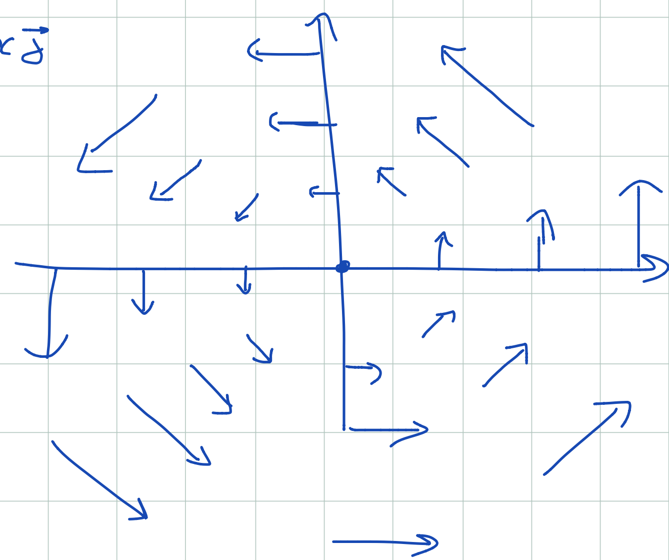
$\vec{F} = \vec{i} + \vec{j}$



$\vec{F} = y\vec{j}$



$\vec{F} = -y\vec{i} + x\vec{j}$



Gradient Vector fields

Given a function $f(x,y,z)$, recall that its gradient
$\nabla f(x,y,z) = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

so it assigns to each point a vector.
If $\vec{F} = \nabla f$ for some function f, we call $\vec{F}$ a gradient vector field & f the potential of the vector field

Examples: • Gravitational force fields

$$\vec{F} = -\frac{GM_1M_2}{\|\vec{r}\|^3}\vec{r} \quad \text{with } f = \frac{M_1M_2G}{\|\vec{r}\|}$$

• Coulomb's law: $\vec{F} = \frac{\varepsilon q_1 q_2}{\|\vec{r}\|^3}\vec{r}$ with $f = -q_1 q_2 \varepsilon / \|\vec{r}\|$

Example: • $\vec{F} = y\vec{i} + x\vec{j}$ is a grad. vec. field with $f = xy$ (how to find f? Integrate $\vec{i}$ component wrt x & $\vec{j}$ comp. wrt y. They must be equal for grad. vec. fields)

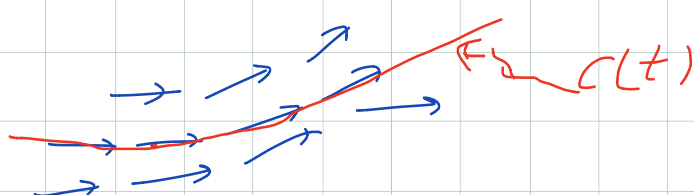
• $\vec{F} = y\vec{i} - x\vec{j}$ is not a grad. vec. field

(Need a fct f so $\frac{\partial f}{\partial x} = y$ & $\frac{\partial f}{\partial y} = -x$)

$$\frac{\partial^2 f}{\partial y \partial x} = 1 \neq \frac{\partial^2 f}{\partial x \partial y} = -1$$

Flow lines

Given a vector field $\vec{F}$, a flow line is a path $\vec{c}(t)$ such that $\vec{F}(\vec{c}(t)) = \vec{c}\,'(t)$



example:

$\vec{F}(x,y) = -y\vec{i} + x\vec{j}$

find a flow line

Need $\vec{c}(t) = (x(t), y(t))$ so that

$\vec{F}(\vec{c}(t)) = \vec{c}\,'(t)$

$$-y(t)\vec{i} + x(t)\vec{j} = x'(t)\vec{i} + y'(t)\vec{j}$$

so we want two functions so that

$$x'(t) = -y(t)$$
$$y'(t) = x(t)$$

$x(t) = \cos(t)$ & $y(t) = \sin(t)$ work

so $\vec{c}(t) = (\cos t, \sin t)$

in fact picking $\vec{c}(t) = (a\cos(t-b), a\sin(t-b))$ would work with any a, b.

Example: Let $\vec{F} = x\vec{i} + 2x\vec{j} + 3y\vec{k}$

find a flow line

$$\vec{c}(t) = (x(t), y(t), z(t))$$

$$\vec{c}\,'(t) = (x'(t), y'(t), z'(t))$$
$$= \vec{F}(\vec{c}(t)) = (x(t), 2x(t), 3y(t))$$

so $x(t) = x'(t) \Longrightarrow x(t) = e^t$

$y'(t) = 2x(t) \Rightarrow y'(t) = 2e^t \Rightarrow y(t) = 2e^t$

$z'(t) = 3y(t) \Rightarrow z'(t) = 6e^t \Rightarrow z(t) = 6e^t$

so $\vec{c}(t) = (e^t, 2e^t, 6e^t)$

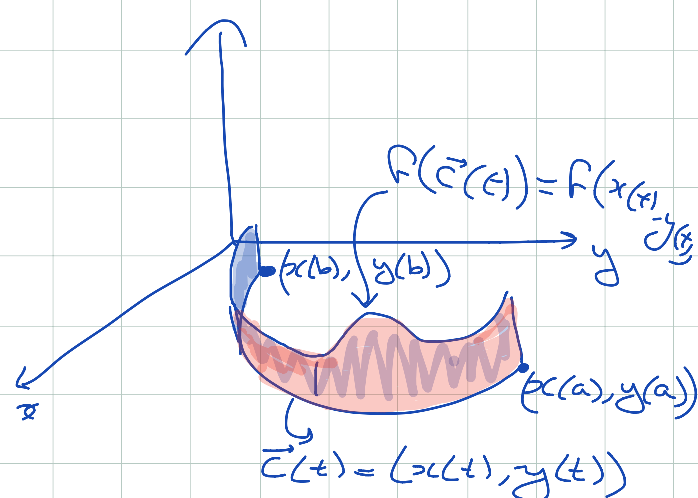
Chapter 7

Integrals over paths and surfaces.

Our eventual goal "Generalize the fund. thm of calculus to Vector calculus". Need to understand integration over paths & surfaces first.

7.1 Path integrals (f is scalar function)

Example (2D) Area of a fence. Let $f: \mathbb{R}^2 \to \mathbb{R}$



$$\text{Area(fence)} = \int_{\vec{c}} f \, ds$$

f: scalar fct.

ds: arc length element

$$ds = \|\vec{c}\,'(t)\| \, dt$$

by definition

$$\int_{\vec{c}} f\, ds := \int_a^b f(x(t), y(t), z(t))\, \|\vec{c}\,'(t)\|\, dt = \int_a^b f(\vec{c}(t))\|\vec{c}\,'(t)\|\, dt$$

Note: need $\vec{c}(t)$ to be cont. Otherwise break $[a,b]$ into pieces.

Example: $\vec{c}(t) = (\cos t, \sin t, t)$ ← helix for $t \in [0, 2\pi]$

$f(x,y,z) = x^2 + y^2 + z$

compute $\int_{\vec{c}} f\, ds$

$$\int_{\vec{c}} f\, ds = \int_a^b f(x(t), y(t), z(t))\, \|\vec{c}\,'(t)\|\, dt$$

where $f(x(t), y(t), z(t)) = x^2(t) + y^2(t) + z(t) = \cos^2 t + \sin^2 t + t = 1 + t$

$\|\vec{c}\,'(t)\| = \|(-\sin t, \cos t, 1)\| = \sqrt{\cos^2 t + \sin^2 t + 1} = \sqrt{2}$

so $\int_{\vec{c}} f\, ds = \int_0^{2\pi} (1+t)\sqrt{2}\, dt = \sqrt{2}(2\pi) + \sqrt{2}(2\pi)^2/2$.

7.2 Line integrals:

Motivation: $W = \text{force} \cdot \text{distance}$ if force is const. & moving in st. line

Over short distances: $W \approx \vec{F} \cdot \Delta \vec{s}$

Total work over long dist. along trajectory C: $W \approx \sum_i \vec{F}(t_i) \cdot (\Delta \vec{s})_i$

$\xrightarrow[\text{limit}]{\text{in the}}$

$$W = \int_{\vec{c}} \vec{F} \cdot d\vec{s} \,.$$

Def'n Let $\vec{F}$ be a vector field on $\mathbb{R}^3$, cont's on $\vec{c}: [a,b] \to \mathbb{R}^3$

The line integral of $\vec{F}$ along $\vec{c}$ is defined as

$$\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int \vec{F}(x(t), y(t), z(t)) \cdot \vec{c}\,'(t)\, dt$$

Example: $\vec{c}(t) = (\cos t, \sin t, t)$, $0 \leq t \leq 2\pi$

$\vec{F}(x,y,z) = x^2 \vec{i} + y^2 \vec{j} + z \vec{k}$

Compute $\int_{\vec{c}} \vec{F} \cdot d\vec{s}$.

Along $\vec{c}$, $\vec{F}(x,y,z) = \vec{F}(\cos t, \sin t, t) = \cos^2 t\, \vec{i} + \sin^2 t\, \vec{j} + t\, \vec{k}$

$$\vec{c}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k}$$

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_0^{2\pi} (\cos^2 t\,\vec{i} + \sin^2 t\,\vec{j} + t\,\vec{k})\cdot(-\sin t\,\vec{i} + \cos t\,\vec{j} + \vec{k})\,dt$$

$$= \int_0^{2\pi} -\cos^2 t \sin t + \cos t \sin^2 t + t \; dt =$$

$$= \int_0^{2\pi} -\cos^2 t \sin t\,dt + \int_0^{2\pi} \cos t \sin^2 t\,dt + \int_0^{2\pi} t\,dt$$

$$= 0 + 0 + \frac{4\pi^2}{2}$$

New notation for the line integral:

$\vec{F} = (P, Q, R)$ & $d\vec{s} = (dx, dy, dz)$

We write $\displaystyle\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_{\vec{c}} P\,dx + Q\,dy + R\,dz$

$$= \int_a^b \left(P\frac{dx}{dt} + Q\frac{dy}{dt} + R\frac{dz}{dt}\right)dt$$

Warning: This is NOT the sum of 3 integrals

It is just another way to write $\int_{\vec{c}} \vec{F}\cdot d\vec{s}$.

We still need to parametrize and express everything in terms of the parameter.

Example: In the previous example

$\vec{c}(t) = (\cos t, \sin t, t)$, $0 \le t \le 2\pi$

$\vec{F}(x,y,z) = x^2\,\vec{i} + y^2\,\vec{j} + z\,\vec{k}$

So $\int_{\vec{c}} \vec{F} \cdot d\vec{s} = \int_{\vec{c}} x^2\, dx + y^2 dy + z\, dz$

$$= \int_0^{2\pi} \left(x^2(t) \frac{dx}{dt} + y^2(t) \frac{dy}{dt} + z(t) \frac{dz}{dt} \right) dt$$

$$= \int_0^{2\pi} \left(\cos^2 t \cdot (-\sin t)\, dt + \sin^2 t \cos t\, dt + t \right) dt$$

$$= 0 + 0 + \frac{4\pi^2}{2} \quad (\text{same as before})$$

<u>Example</u>:

evaluate $\int_{\vec{c}} x^2 dx + xy\, dy$

where $\vec{c}(t) = (t, t^2)$, $0 \le t \le 1$

$$\int_{\vec{c}} x^2 dx + xy\, dy = \int_0^1 \left(x^2 \frac{dx}{dt} + xy \frac{dy}{dt} \right) dt = \int_0^1 (t^2 \cdot 1 + t^3 \cdot 2t)\, dt$$

$$= \int_0^1 t^2 + 2t^4\, dt = \frac{1}{3} + \frac{2}{5} = \frac{11}{15}.$$

<u>Remark</u>: Line integrals are independent of the parametrization as long as the parametrization is orientation preserving.

<u>ex</u> the curve $y = x^3$ from $(0,0)$ to $(1,1)$ can be parametrized as $\vec{c}(t) = (t, t^3)$ $\quad 0 \le t \le 1$

or as $\vec{p}(\theta) = (\sin\theta, \sin^3\theta)$, $0 \le \theta \le \pi/2$

Let $\vec{F} = x\vec{\imath} + y\vec{\jmath}$

$$\int_{\vec{c}} \vec{F}\cdot d\vec{s} = \int_0^1 (t\vec{\imath} + t^2\vec{\jmath})\cdot \underbrace{\vec{c}\,'(t)}_{\vec{\imath} + 3t^2\vec{\jmath}}\,dt$$

$$= \int_0^{\pi/2} (\sin\theta\,\vec{\imath} + \sin^3\theta\,\vec{\jmath})\,\underbrace{\vec{p}\,'(\theta)}_{\cos\theta\,\vec{\imath} + 3\sin^2\theta\cos\theta\,\vec{\jmath}}\,d\theta$$

Exercise: check this